Elastic waves in a slinky

W.E. Bailey, APAM/MSE EN1102

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Differential spring element

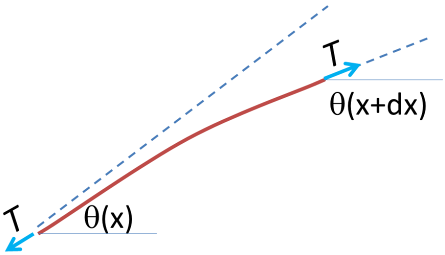

Figure: Differential length dx of spring under tension T with curvature

$$\theta = \theta(x)$$

is not a constant.

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$$\frac{\partial y(x)}{\partial x} \simeq \theta(x)$$

Differential spring element: Newton's laws in y

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Small θ: approximate $\sin\theta \simeq \theta$

$$T\left(\theta(x + dx) - \theta(x)\right) = m\frac{\partial^2 y}{\partial t^2}$$

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This is a partial differential equation! (PDE).
There are partial derivatives in x, t.

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where k is the "wavenumber," in m^{-1}, and ω is the circular frequency in s^{-1}. These quantities correspond to wavelength, period:

$$\lambda = \frac{2\pi}{k} \qquad \tau = \frac{2\pi}{\omega}$$

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v is the velocity at ω

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$$T = k\Delta L$$

(k: spring constant)

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Looks like what? (see)

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very weird! You can't "tune" a slinky! (guitar strings are different)