MSAE3111, Thermodynamics and Statistical Mechanics

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Sep 15, 2011 (#4)

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- **Question**: how much more probable for 1 to be occupied than 2 at temperature T?

We divide the universe

- Two regions:

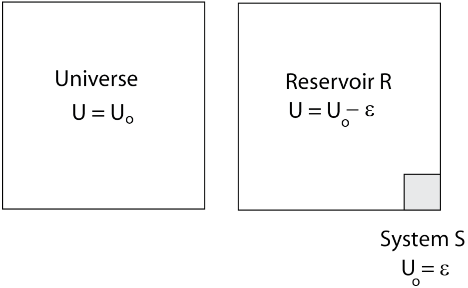

Figure: Universe: tiny system, big reservoir.

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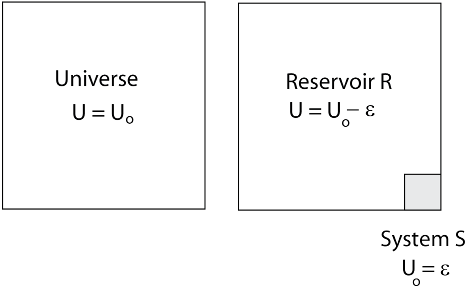

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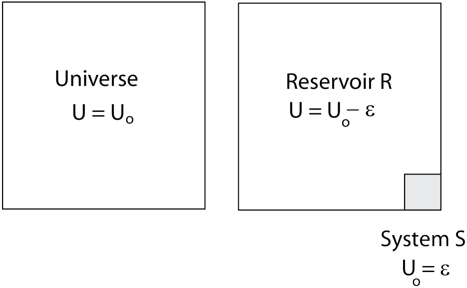

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  $\Rightarrow U_{\mathbb{R}} = U_0 - \epsilon$

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- Total multiplicity is multiplicity of reservoir $\mathbb{R}$

$$g_{tot} = g^{\mathbb{R}} \tag{2}$$

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- In terms of reservoir $\mathbb{R}$

$$\frac{P(\epsilon_1)}{P(\epsilon_2)} = \frac{g^{\mathbb{R}}(U_o - \epsilon_1)}{g^{\mathbb{R}}(U_o - \epsilon_2)} \tag{4}$$

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- From $\sigma = \ln g$,

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- expand $\sigma(U_o - \epsilon)$, $\epsilon \ll U_o$–system is tiny

$$\sigma(U_o - \epsilon) \simeq \sigma(U_o) - \epsilon \frac{\partial \sigma}{\partial \epsilon}(U_o) \ldots \tag{6}$$

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- Finally:

$$\frac{P(E_2)}{P(E_1)} = e^{\frac{-\Delta E}{k_B T}} \tag{9}$$

Two-level system

- See plot:

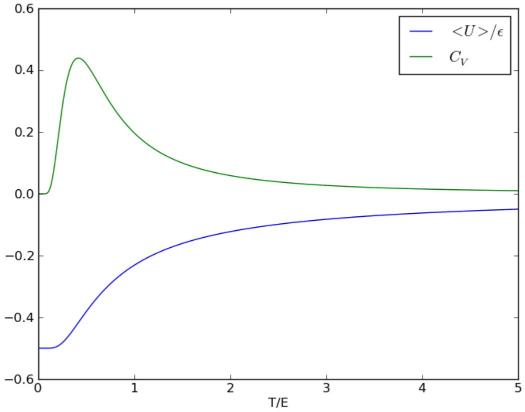

Figure: Heat capacity and average energy for a two-level system: spacing ϵ at temperature T.